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Śrîpati: An Eleventh-Century Indian Mathematician

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Śrîpati (fl. A.D. 1039–1056) is best known for his writings on astronomy, arithmetic, mensuration, and algebra. This article discusses Śrîpati's arithmetic, the *Gaṇitatilaka,* as well as the arithmetical and algebraic chapters of the *Siddhântaśekhara*. In addition to discussing the kinds of problems considered by Śrîpati and the techniques he used to solve them, the article considers the sources upon which Śrîpati drew. A glossary of Indian treatises and technical terms is provided. © 1985 Academic Press, Inc.

Śrîpati (actif vers 1039–1056 ap. J.C.) est surtout connu pour ses écrits sur l'astronomie, l'arithmétique, le toisé, et l'algèbre. Dans cet article, nous abordons l'arithmétique de Śrîpati, le *Gaṇitatilaka,* de même que les chapitres arithmétiques et algébriques de son *Siddhântaśekhara*. En plus d'aborder les types de problèmes étudiés par Śrîpati et les techniques qu'il a employées pour les résoudre, nous exminons aussi les sources auxquelles Śrîpati fait appel. Un glossaire des traités et des termes techniques indiens complète cet article. © 1985 Academic Press, Inc.

Śrîpati, der zwischen 1039 und 1056 wirkte, ist vor allem durch seine Schriften über Astronomie, Arithmetik, Mensuration und Algebra bekannt. In diesem Beitrag werden seine Arithmetik, die *Gaṇitatilaka,* und die arithmetischen und algebraischen Kapitel aus *Siddhântaśekhara* behandelt. Neben der Art der Probleme, die Śrîpati studierte, und den von ihm verwendeten Lösungsmethoden werden auch die von ihm benutzten Quellen betrachtet. Der Artikel schließt mit einem Verzeichnis der indischen Titel und Fachtermini. © 1985 Academic Press, Inc.

INTRODUCTION

Śrîpati was the most prominent Indian mathematician of the 11th century A.D. [Dvivedi 1892, 29]. He gave the date śaka 961 (i.e., A.D. 1039) for his work *Dhîkoṭidakaraṇa*[1] and śaka 978 (i.e., A.D. 1056) for the *Dhruvamânasa*. So it is certain that Śrîpati's period of activity includes A.D. 1039–1056.

From Śrîpati's own works (including the *Jâtakapaddhati*) and the testimony of Mahâdeva in his commentary on the *Jyautiṣaratnamâlâ,* one learns that Śrîpati was the son of Nâgadeva and the grandson of Keśavabhaṭṭa, a Brahmin by caste and a kâśyapa by lineage. According to Śrîpati's own statement in the *Dhruvamânasa,* he composed that work at Rohiṇîkhaṇḍa. The work that definitely locates Śrîpati's home in Mahârâṣṭra is his Marathi commentary on the *Jyautiṣaratnamâlâ*. Panse [1957] has identified Rohiṇîkhaṇḍa with Rohinîkheḍa in the Buldhana District of that state.

Śrîpati's *Siddhântaśekhara* is an astronomical work in twenty chapters, among which are the vyaktagaṇitâdhyâya (chapter on arithmetic and mensuration) and avyaktagaṇitâdhyâya (chapter on algebra). In addition to the mathematical chap-

[1] A *Glossary* of terms may be found at the end of the article.

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ters of the *Siddhântaśekhara*, Śrîpati is the author of an arithmetic known as *Gaṇitatilaka* which was commented upon by Siṁhatilaka Sûrî (A.D. 1275 ?). While the mathematical chapters of the *Siddhântaśekhara* contain only rules, the *Gaṇitatilaka* contains as well numerous fanciful problems as exercises that illustrate these rules. The existence of another mathematical work of Śrîpati, a *Bîjagaṇita* (algebra), is as yet known only from its being referred to by Munîśvara Viśvarûpa (A.D. 1608) in his commentary *Lîlâvatîvivṛti* [Dvivedi 1892, 29–30].

THE *GAṆITATILAKA*

The only edition of the *Gaṇitatilaka* comprises 125 verses, of which two (the sûtra on probhâgajâti with its uddeŝaka) are missing from the manuscript. The book discusses in order: terminology (containing numeration by tens up to 10^{17}, and the measures used); the eight operations with integers, fractions and zero; reduction of five standard classes of fractions; problems related to the solution of linear, quadratic, and radical equations in one unknown; the rule of three and inverse rule of three; the rule of five and dependent rules of barter and sale of living animals; simple interest; usury; the rule for conversion of a number of bonds into a single one (ekapatrîkaraṇa); and the rule for equating shares of capital given by different persons for unequal periods of time (samîkaraṇa).

It is to be noted that the *Gaṇitatilaka* does not contain progressions, mensuration of plane and solid figures, or shadow reckoning, which are common topics of Hindu arithmetic. So its editor Kapadia [1937] had rightly stated that the unique manuscript at his disposal was incomplete. It is noteworthy that most of the sûtras of the *Gaṇitatilaka* are included among verses 1–18 of the vyaktagaṇitâdhyâya of the *Siddhântaśekhara*. It is almost certain, therefore, that the lost portion of the *Gaṇitatilaka* consisted essentially of verses 19–55 of the vyaktagaṇitâdhyâya with appropriate uddeśakas. It is remarkable that, except for a single reference to the Hindu deity Maheśvara [Kapadia 1937, 6], the extant fragment of the *Gaṇitatilaka* is quite secular in nature. A reference to a female student [Kapadia 1937, 67] indicates that the study of mathematics by women was in vogue during Śrîpati's time.

In writing the *Gaṇitatilaka*, Śrîpati closely followed the *Triśatî* of Śrîdhara and borrowed many things from it. The order and content of the sûtras of the *Gaṇitatilaka* with a few exceptions (discussed below) seem to be based on the rules of the *Triśatî*. Most of Śrîdhara's exercises (25–62) are also included as uddeśakas among verses 56–122 of the *Gaṇitatilaka*, sometimes word for word.

The earliest known treatment of the addition and subtraction of whole numbers in any Indian work is to be found in the *Gaṇitatilaka*. The sûtra for addition in the *Gaṇitatilaka* is:

> As there is the addition of a number to [that of] its own side by the krama [method], so [the same] is to be performed in the process of addition, by the utkrama [method]. [Kapadia 1937, 3]

According to Siṁhatilaka, there is successive columnwise downward addition of numbers, starting from the topmost in the krama method; under the utkṛama method, there is successive columnwise upward addition of numbers starting from the lowest.

The sûtra for subtraction in the *Gaṇitatilaka* is:

> And also for the acquisition of the remainder in the process of subtraction, the subtraction is carried out by that [previous] method. [Kapadia 1937, 4]

According to Siṁhatilaka, there are two methods of subtraction, namely, placing the subtrahend below the minuend and vice versa.

An interesting feature of the author's treatment of multiplication in the *Gaṇitatilaka* is the following garland problem

> [What is the number obtained] when 12987013 is multiplied by 77 so that it may represent the clear, sparkling and spherical pearls forming an ornament here for the neck of Maheśvara? [Kapadia 1937, 6]

The answer to this problem is 1000000001. Such problems occur in abundance in Mahâvîra's *Gaṇitasârasaṁgraha* (ii, 10–17) [Rangacarya 1912]. They are not found in any other Indian work.

The sûtra for division of fractions contained in the *Gaṇitatilaka* is:

> On making interchange of the numerator and denominator of the divisor, [and] cross-reduction in that state, afterwards the act of the pairwise multiplication of the numerators and the denominators is always to be made by him who desires the division [of fractions]. [Kapadia 1937, 21]

Here the author has recommended the use of cross-reduction for shortening the work of division although no other Indian author recommended cross-reduction in the case of division; Mahâvîra, in his *Gaṇitasârasaṁgraha* (iii, 2), referred to cross-reduction in order to shorten the work of multiplication.

The sûtra of the *Gaṇitatilaka* for operations with zero is:

> On addition of an additive, zero becomes equal to the additive; on subtraction or addition of zero, a quantity is unchanged; and on multiplication by zero, [a quantity becomes] zero. Zero results on division [of a quantity] by zero as well as on division of zero [by zero]. On squaring zero, it should be zero, and the cube [of zero should also be zero]. [Kapadia 1937, 29]

(See also the discussion of avyaktagaṇita, below.)

It is to be noted that the author made a distinction between $0 \div a$ and $a \div 0$, though in each case he gave the result to be zero. The other interesting features of the rule are that the indeterminate nature of 0/0 and the infinite nature of a/0 (where $a \neq 0$) were not recognized here, the results being simply stated as $0 \div 0 = 0$ and $a \div 0 = 0$.

It is also of interest to note that though Brahmagupta used the result $a/0 = \infty$ in the astronomical part of his *Brâhmasphuṭasiddhânta* (vii, 14) [Dvivedi 1902], he stated in the algebraic chapter (xviii, 36) that "a negative or positive (number) divided by zero is 'taccheda,' or zero divided by a negative or positive [number]." He also wrote that "zero divided by zero is zero" (xviii, 35). One must conclude, therefore, that he was inconsistent, following one tradition in mathematics, another, contradictory, tradition in astronomy. Mahâvîra gave the result of $a \div 0$ to be a [*Gaṇitasârasaṁgraha*, i, 49).

In the *Gaṇitatilaka,* rules were given for the reduction of five standard classes of fractions (bhâga, prabhâga, bhâgânubandha, bhâgâpavâha, and vallîsavarṇana) [Kapadia 1937, 30–41] while Śrîdhara considered two additional classes, bhâga-bhâga and bhâga-mâtṛ, in his *Pâṭîgaṇita* (Rules 38, 42) [Shukla 1959] *and Triśatî* [Dvivedi 1899, 11–12]. Mahâvîra also considered these two classes in his *Gaṇita-sârasaṁgraha* (iii, 99, 138) and remarked that there might be twenty-six variations of the last type.

Śrîpati gave the rule for the reduction of fractions to a common denominator as follows:

> One should multiply the numerator and denominator [of each of a given pair of fractions] reciprocally by the two denominators for bringing equality in the denominators [of these fractions]. [Kapadia 1937, 30]

Instead of using the lowest common multiple of the denominators as the common denominator, as had Mahâvîra (iii, 56) and the *Bakshâlî Manuscript* (e.g., folio 1, verso; folio 17, recto), Śrîpati chose the product of the two denominators of a pair of fractions as the common denominator. Here he followed the corresponding rule of the *Triśatî* [Dvivedi 1899, 10] based on Brahmagupta's rule (xii, 2). Another method Śrîpati ignored here is the multiplication of the numerator and denominator of one fraction by the denominator of another which had previously been divided by the common factor; this method was stated by Mahâvîra in his *Gaṇitasârasaṁgraha* (iii, 55) and by Śrîdhara in his *Pâṭîgaṇita* (Rule 36).

Śrîpati's treatment of linear, quadratic, and radical equations in the *Gaṇitatilaka* [Kapadia 1937, 41–65] differs from those given by Śrîdhara in his *Triśatî* [Dvivedi 1899, 13] or *Pâṭîgaṇita* (Rules 74–77) and by Mahâvîra in his *Gaṇitasâra-saṁgraha* (iv, 4–70). Śrîpati's rule for dṛśyajâti is:

> In the jâti called the dṛśya, one should divide the known quantity by unity lessened by the sum of the [given] parts. [Kapadia 1937, 41]

This involves the solution $x = a/(1 - b)$ of the equation $x - bx = a$, where x is the total unknown, a is the given part, and b is the sum of the parts removed. Śrîdhara, in the *Triśatî* [Dvivedi 1899, 13] and *Pâṭîgaṇita* (Rule 74), calls this stambhoddeśa or stambhajâti, while Mahâvîra (iv, 4) calls it bhâgajâti.

Śrîpati [Kapadia 1937, 44] and Mahâvîra (iv, 4) gave different rules for dealing with linear problems involving remainders (śeṣajâti). Śrîpati's rule is:

> The known quantity is to be divided by the product of the denominators lessened by the [respective] numerators, [the said product being previously] divided by the product of the denominators.

This gives the solution $x = a/((y_1 - x_1)(y_2 - x_2) \ldots /y_1y_2 \ldots)$ of the equation $x - (x_1/y_1)x - (x_2/y_2)(x - (x_1/y_1)x) - \ldots = a$, where a is the known quantity and $x_1/y_1, x_2/y_2, \ldots$ are the known fractional parts of the successive remainders. Mahâvîra gave the rule in the form: $x = a/(1 - b_1)(1 - b_2)(1 - b_3) \ldots$, where

$b_1, b_2, \ldots$ are fractional parts removed from successive remainders. Śrîdhara gives a similar rule under mûlâdeśeṣajati in his *Pâṭîgaṇita* (Rule 75).

The stambhoddeśa is the only sûtra in the *Triśatî* dealing with problems involving equations. Śrîpati borrowed two of his examples in the *Gaṇitatilaka* from those of the *Triśatî* in its stambhoddeśa (Exercises 25, 27); he gave the first under dṛśyajâti as verse 56, and the second as the first example under śeṣajâti.

Śrîpati's rule for viśleṣajâti is:

> In the viśleṣajâti after the subtraction of the lesser number from the greater, the remaining operation is as stated [under the bhâgajâti]. Thereafter, having subtracted the sum of the parts from unity, one should divide the [given] part of the known number by the [obtained] remainder. [Kapadia 1937, 46]

Śrîdhara gave the rule under viśeṣajâti in his *Pâṭîgaṇita* (Rule 74). Mahâvîra did not differentiate between bhâgajâti and viśleṣajâti, placing problems on viśleṣajâti under bhâgajâti (iv, exercises 23–27).

The śeṣamûlajâti of Śrîpati [Kapadia 1937, 48] and of Mahâvîra (iv, 40) is Śrîdhara's mûlâdiśeṣajâti (Rule 75), which also contains a rule similar to the śeṣajâti. The mûlâgrabhâgajâti of Śrîpati [Kapadia 1937, 50–51] corresponds to the mûlajâti of Mahâvîra (iv, 33) and the bhâgamûlâgrajâti of Śrîdhara (Rule 76). Similarly, the ubhayâgradṛśyajâti of Śrîpati [Kapadia 1937, 54] is the dviragraśeṣamûlajâti of Mahâvîra (iv, 47) and the ubhayâgramûlaśeṣajâti of Śrîdhara (Rule 77). Further, the bhinnabhâgadṛśyajâti of Śrîpati is the bhinnadṛśyajâti of Mahâvîra (iv, 69); Śrîdhara did not give a corresponding rule. Śrîpati's sûtra for this jâti is:

> Its measure [i.e., the measure of the post] should be [obtained] after unity is lessened by the visible portion, and subsequently [the remainder is] divided by the product of the given parts. [Kapadia 1937, 58]

Śrîpati's bhâgamûlajâti (for which Śrîdhara gave no corresponding rule) is:

> The square [root] is extracted from the known quantity [which has been successively] multiplied by four, divided by the [given] part, and [then] united with the square of the [given] coefficient of the root related to [the known quantity] itself. [That square-root] united with the [given] coefficient of the root is halved, [the resulting half is] squared, and [the obtained square is] multiplied by the [given] part. The measure of the herd should be here. [Kapadia 1937, 60]

This gives the solution

$$x = \{[b_1 + \sqrt{b_1^2 + 4(a/b_2)}]/2\}^2 \cdot b_2$$

of the equation $x - b_1\sqrt{b_2 x} = a$.

Mahâvîra gave the same rule (iv, 52) and also a simplified form (iv, 51):

$$x = \{(b_1 b_2/2) + \sqrt{(b_1 b_2/2)^2 + ab_2}\}^2 \div b_2.$$

Śrîdhara also gave no rule corresponding to Śrîpati's sûtra for hînavargajâti, which is as follows:

> [The fraction] that has its denominator brought to the top by its numerator [which itself goes down] is [written] here in two places. That one is multiplied by the subtrahend, [the product

is] united with the square of the half of the other, [the resulting sum is then] lessened by the known quantity. The [square] root of it [i.e., the said remainder] is united with the subtrahend and one-half of the other [number], [and the resulting sum is] divided by the [given] fraction. The quotient is the required result. [Kapadia 1937, 62]

This gives the solution

$$x = (p + (b/2a) + \sqrt{(b/a) \cdot p + (b/2a)^2 - q}) \div (a/b)$$

of the equation $x - ((a/b) \cdot x - p)^2 = q$, where a and b are integers and p and q are any numbers.

Mahâvîra's rule corresponding to the hînavargajâti, under his aṁśavargajâti (iv, 61), differs in that it also treats another variety of problem where the equation involved is of the type $x - ((a/b) \cdot x + p)^2 = q$, whose solution is

$$x = ((b/2a) - p - \sqrt{(b/2a)^2 - (b/a) \cdot p - q}) \div (a/b).$$

The sûtra on the rule of three in the *Gaṇitatilaka* is:

The pramâṇa [argument] and the abhîpsâ [requisition] are [respectively] at the beginning and end; and the phala [fruit] is in the middle; [and so] a different species is made [in between the first two which are of the same species]. Having multiplied the phala by the samîcchâ [requisition], one should divide it by the pramâṇa. The operation is reversed in the inverse rule of three. [Kapadia 1937, 68]

Śrîdhara, under compound proportion, gave sûtras for rules of five, seven, and nine terms in his *Triśatî* [Dvivedi 1899, 19] or *Pâṭîgaṇita* (Rule 45). Śrîpati gave only the rule of five [Kapadia 1937, 74], but provided examples for rules of seven, nine, and eleven terms [Kapadia 1937, 78–80], the examples for the first two being borrowed from the *Triśatî* (exercises 51, 52). Siṁhatilaka [Kapadia 1937, 78] has explained the absence of these rules from the *Gaṇitatilaka* by stating that the procedure of the rule of five was equally applicable to the other cases.

It is also noteworthy that, whereas Śrîdhara gave a separate rule for suvarṇa-gaṇita (calculations pertaining to gold) both in his *Triśatî* [Dvivedi 1899, 19] and *Pâṭîgaṇita* (Rule 47), Śrîpati considered problems related to this rule under the rule of three and inverse rule of three [Kapadia 1937, 73–74].

Interest problems are treated in the *Gaṇitatilaka* by a sûtra borrowed word for word from Brahmagupta's *Brâhmasphuṭasiddhânta* (xii, 14) [Dvivedi 1902] and a sûtra which seems to be Śrîpati's own. The original sûtra of the author is:

[Scholars] know that the fruit [meaning interest] is arrived at by multiplying commodity [meaning the capital] by the [number of] months, multiplying [the product] again by the increase [meaning the rate of interest], and dividing [the product] by one hundred. [Kapadia 1937, 85]

It is to be noted that two examples (vv, 112, 114) are the same as exercises 52 and 54 in the *Pâṭîgaṇita* of Śrîdhara.

Śrîpati's rule for ekapatrîkaraṇa is remarkable because he followed Śrîdhara (*Pâṭîgaṇita,* Rule 51) despite the existence of a simplified rule in the work of Âryabhaṭa II [Dvivedi 1910]:

The [average] passed time is [known] when the sum of the fruits of the passed periods of time are divided by the sum of the monthly increases. Also, the fruit for one hundred is [known] in the "ekapatrîvidhâna" when that [sum of monthly profits] is multiplied by one hundred and [the obtained product] is divided by the sum of the wealths. [Kapadia 1937, 86]

Here a method for the conversion of several bonds into an equivalent one is described. In modern mathematical language, the rule is as follows.

Let the capital, rate of interest, and the time elapsed for each of n given bonds be $P_1, P_2, \ldots, P_n$, $i_1, i_2, \ldots, i_n$ percent per month, and $t_1, t_2, \ldots, t_n$, respectively. Then the time, t, in months, elapsed and the rate of interest per month, i, for the equivalent single bond are given by

$$t = \frac{P_1t_1i_1/100 + P_2t_2i_2/100 + \cdots + P_nt_ni_n/100}{P_1i_1/100 + P_2i_2/100 + \cdots + P_ni_n/100},$$

$$i = \frac{P_1i_1/100 + P_2i_2/100 + \cdots P_ni_n/100}{P_1 + P_2 + \cdots + P_n} \cdot 100 \text{ percent.}$$

Âryabhaṭa II (xv, 33) put these results in the simplified forms

$$t = (P_1t_1i_1 + P_2t_2i_2 + \cdots + P_nt_ni_n)/(P_1i_1 + P_2i_2 + \cdots + P_ni_n),$$

$$i = (P_1i_1 + P_2i_2 + \cdots + P_ni_n)/(P_1 + P_2 + \cdots + P_n).$$

Mahâvîra (vi, 77, 77½) also gave rules for finding the average common time and the average common interest. His rules are more general in that he did not use interest percent.

Śrîpati's rule for samîkaraṇa seems to be of his making.

The parts of the employed quantity [i.e., the capital] surely are [known] by means of the multiplication of the unblended wealth [i.e., the principal] by the interchanged numerators and denominators [i.e., the two times on the one side and the pramâṇarâśi and the phala on the other], [taking] the profit [i.e., the accrued interest] separately as unity [for each part of the capital], and the division [of the aforesaid product] by their sum [i.e., the sum of the parts of the capital each yielding unit interest]. [Kapadia 1937, 90]

This is a rule for equating shares of capital given by different grâhakas for unequal periods of time. In modern mathematical language, the rule may be interpreted as follows: Let $P_1, P_2, \ldots, P_n$ be the parts of the capital, P, yielding the same interest in $m_1, m_2, \ldots, m_n$ months at rates of interest $i_1, i_2, \ldots, i_n$, each of which is earned by sums $c_1, c_2, \ldots, c_n$ for $t_1, t_2, \ldots, t_n$ months, respectively. Then the parts P_r of the capital P yielding the same interest are given by $P_r = (t_rc_r/m_ri_r)/(t_1c_1/m_1i_1 + t_2c_2/m_2i_2 + \cdots + t_nc_n/m_ni_n)$, where $r = 1, 2, \ldots, n$.

THE *VYAKTAGAṆITÂDHYÂYA*

The arithmetical chapter of the *Siddhântaśekhara* consists of fifty-five verses. The meager manuscript evidence did not permit the editor, B. Misra, to establish the correct text of some of the verses; thus four of the verses (10, 25, 27, and 30) are incomplete and two (11, 26) are too obscure to be interpreted. The first verse is introductory in nature and states that in order to be a distinguished gaṇaka (calculator) one will have to master the twenty parikarmas (logistics) and the eight vyavahâras (determinations) contained in this chapter. The remainder of the work contains, in order, operations with integers (here addition and subtraction have been omitted as if presupposed) and fractions (here square root, cube, and cube root have been omitted), the reduction of five classes of fractions, the rule of three and inverse rule of three; the rule of five with dependent rules of the barter and sale of living animals, simple interest, usury, partnership, progressions, mensuration of plane and solid figures, and shadow reckoning. It is noteworthy that of verses 1–18, all except six (namely, 1, 3, 4, 8, 11, 12) appear in the *Gaṇitatilaka*. So it is almost certain that the lost portion of the *Gaṇitatilaka* consisted essentially of verses 19–55 with proper exercises.

The rule of division in the *Siddhântaśekhara* differs from that in the *Gaṇitatilaka:*

> One should place the divisor below the dividend. The quotient [is that] multiplied by which [the divisor] is removed from the dividend. So, it should originate from the diminution of the dividend during the process of division [proceeding] in reverse order from the end [of the dividend]. (xiii, 3)

This is similar to the rule of division given by Âryabhaṭa II (xv, 4), whereas the corresponding rule in the *Gaṇitatilaka* [Kapadia 1937, 6] is similar to the rule in the *Triśatî* [Dvivedi 1899, 4] and *Pâṭîgaṇita* (Rule 22) and that in the *Gaṇitasârasaṁgraha* (ii, 19).

While Śrîpati had described three methods for squaring [Kapadia 1937, 7–8] and four methods for cubing [Kapadia 1937, 11–12] in the *Gaṇitatilaka*, he gave only definitions of squares and cubes in the *Siddhântaśekhara:*

> A square is the [result of the] multiplication of two equal quantities. A cube is the [result of the] multiplication of three equal quantities. [Scholars] say that a square is a four-sided area. A cube should be of twelve [equal] sides; it [is also called] vṛnda. (xiii, 4)

Under the topic of the addition and subtraction of fractions (xiii, 8), Śrîpati gave a rule for the reduction of fractions to a common denominator. While the rule for prabhâgajâti is missing from the *Gaṇitatilaka*, it appears in the *Siddhântaśekhara:*

> In the prabhâgajâti, for savarṇana, there should be [mutual] multiplication of the denominators and of the numerators. (xiii, 11, first part)

The second part of 11, as printed, is obscure. But if we amend adhaścidà to adhaśchidâ, it may be translated as:

> One should multiply the preceding numerator and denominator by the lower denominator. [One should then apply] the lower numerator, [which may be] positive or negative, to the former numerator [i.e., the upper numerator multiplied by the lower denominator].

This is a sûtra for vallîsavarṇanajâti which appears more explicitly in the *Gaṇitatilaka* [Kapadia 1937, 39].

The second hemistich of 10 is incomplete as preserved; but it is almost certain that it dealt with bhâgânubandha and bhâgâpavâha classes. Therefore, in the

Siddhântaśekhara, the same five classes of fractions are treated as in the *Ganitatilaka*.

The treatment of arithmetic progressions in the *Siddhântaśekhara* is in the tradition of the *Brâhmasphuṭasiddhânta*. The only exceptions are Śrîpati's rules for the first term, common difference, and the number of terms of an arithmetic series; these are not found in the *Brâhmasphuṭasiddhânta*. The rules of the *Siddhântaśekhara* for the first term and the common difference are:

> The first term should result when the sum of the series is divided by the number of terms, and [the quotient is] diminished by half the common difference multiplied by the number of terms less one. The common difference should result when the sum of the series is diminished by the product of the number of terms and the first term, and [the residue is] divided by half the square of the number of terms diminished by the number of terms. (xiii, 23)

This gives $a = (S/n) - (d/2)(n - 1)$ and $d = (S - an)/((n^2 - n)/2)$, where S is the sum of the series; a, the first term; d, the common difference; and n, the number of terms.

Mahâvîra (vi, 292) is the first Indian author to have stated these rules. The formulas he used are $a = (S/n) - ((n - 1)/2) \cdot d$ and $d = ((S/n) - a)/((n - 1)/2)$. Similar rules are found in all later works.

Rules for finding the number of terms of an arithmetic series are found in all Indian works. But the rule of the Siddhântaśekhara differs from the rest, the others being mere variations of each other. Śrîpati's rule is:

> You should add the first term [which has been previously] diminished by half the common difference, [the residue being then] divided by the common difference [and the quotient] squared, to the sum of the series divided by half the common difference. [Scholars] say that the square-root of that [sum] diminished by the square-root of the previous quantity is the period. (xiii, 24)

Interpreted in modern mathematical notations, this rule gives

$$n = \sqrt{S/(d/2) + [(a - (d/2))/d]^2} - \sqrt{[(a - (d/2))/d]^2},$$

while Brahmagupta's rule (xii, 18) is expressed by

$$n = (\sqrt{8dS + (2a - d)^2} - (2a - d))/2d,$$

and Mahâvîra's (vi, 294) by

$$n = [((S/n) - t_1)/(d/2)] + 1,$$

where S/n is called the lâbha. Other Indian rules are either the same as Brahmagupta's or variations of it.

Śrîpati's rule for the summation of a series in geometric progression is given in the incomplete verse (xiii, 25):

> In case of an even period, when it is halved, you should place square [by its side]; in case of an odd period, when it is lessened by one, [place] multiplier [by its side]. The result obtained through [alternate] multiplication and squaring in the upward order, is lessened by one, [the remainder is divided by the multiplier decreased by one; the quotient], multiplied [by the first term, is the sum].

Though the last line of the verse is incomplete, there is no difficulty in understanding the rule as it is the same as that of Bhâskara II's *Lîlâvatî* (vs. 130). To describe the rule in modern notation, let a, r, and n be, respectively, the first term, common ratio, and the number of terms of a series in geometric progression. If n is even, write $n/2$ in one place and write "square" by its side; if n is odd, write $n - 1$ and "multiplier" by its side. Repeat the same process for $n/2$ or $n - 1$, as the case may be, according as this number is even or odd. Continue the process until the end (i.e., until the number 1 is obtained). Now, commencing from the bottom, proceed upward multiplying by r at places marked "multiplier," and squaring the previous expression at places marked "square." If N is the number finally obtained, then the sum of the series is $a(N - 1)/(r - 1)$.

Thus Śrîpati did not use the formula $S_n = a(r^n - 1)/(r - 1)$ for summing the series. Instead, he arrived at the value of r^n by a peculiar process introduced, in all probability, because the appropriate notation and symbols were not available. Pṛthudaka [Colebrooke 1817, 291] gave the same rule as Śrîpati for the summation of series which are "geometrically twofold," and made quite a general application of the rule. In his *Pâṭîgaṇita* (Rules 94, 95), Śrîdhara gave almost the same rule. Brahmagupta and Âryabhaṭa I did not consider the summation of geometric series, whereas Mahâvîra in his *Gaṇitasârasaṁgraha* gave two rules. In one of these rules (ii, 93), he called the $(n + 1)$st term of the geometric series the guṇadhana; he then gave the sum in the usual form. In another sûtra (ii, 94), Mahâvîra gave the same rule as that of Śrîpati, instructing the reader to mark by 0 and 1, respectively, the even value which is halved and the odd value from which 1 is subtracted.

It is noteworthy that Śrîpati excluded from his treatment the following topics, which are regarded as the highlights of the Indian mathematical literature on arithmetic series: (1) "vyutkalita" (summation of a series after a certain number of initial terms has been cut off), a theory which occupied the attention of Âryabhaṭa I (ii, 19) and Mahâvîra (Chap. II); (2) the treatment of series with a fractional number of terms, discussed by Mahâvîra in his *Gaṇitasârasaṁgraha* (Chap. iii) and by the author of the *Bakshâlî Manuscript* (e.g., on folio 5, recto); (3) diagrammatic representation of series, described by Śrîdhara in his *Pâṭîgaṇita* (Rules 79ff.).

In order and content, Śrîpati's mensuration of plane and solid figures is almost the same as that of Brahmagupta; most of Śrîpati's rules are to be found in the *Brâhmasphuṭasiddhânta* (xii, 21–51). However, Śrîpati introduced some new rules, eliminated Brahmagupta's crude value of 3 for π, and substituted for Brahmagupta's rule for the volume of the frustum of a rectangular pyramid (xii, 45–46) the corresponding rule of Âryabhaṭa II (xv, 106). The characteristic features of Śrîpati's mensuration are briefly described below:

Śrîpati's kṣetragaṇita opens with the two verses (xiii, 26–27) of which the first hemistich of the first is obscure (it is not even possible to determine whether it is related to progressions or to mensuration of plane figures), while the second hemistich of the second is missing. The second hemistich of the first verse and the first hemistich of the second verse may, however, be translated as follows:

> In a quadrilateral, [the sum] of all [the other sides] should be greater than a non-curved side. If the sum of the other sides [of a quadrilateral] is less than or equal to [the remaining side], that [figure] is to be known by intelligent persons as a non-area.

Śrîdhara's rule is more explicit:

> [In rectilinear figures], the sum of all the sides except one is neither equal to nor less than the side excepted, because a curved path is neither less than nor equal to a straight path. [*Pâṭîgaṇita*, Rule 108]

Śrîdhara was the first Indian author to state this rule, which also appears in the *Mahâsiddhânta* (xv, 64).

Śrîpati's first rule for the area of a quadrilateral is:

> Half the sum of the sides [of a triangle or of a quadrilateral] is placed in four places, [each of these is] lessened separately by the sides in order, [the lessened quantities being then] mutually multiplied; [afterwards the product is] reduced to [its] [square] root. The result is [the measure] of [each of] a triangle and a quadrilateral. (xiii, 28)

The verse gives the correct expression $\sqrt{s(s-a)(s-b)(s-c)}$ for the area of a triangle whose semiperimeter is given by s and sides by a, b, and c, and the related formula $\sqrt{(s-a)(s-b)(s-c)(s-d)}$ for the area of a quadrilateral. The same rule appears in the works of Brahmagupta (xii, 21) and Mahâvîra (vii, 50). In his *Pâṭîgaṇita* (Rule 117) Śrîdhara prescribed the rule for a scalene quadrilateral. Âryabhaṭa II (xv, 69) pointed out that the rule is accurate in the case of a triangle, but not for a quadrilateral. Śrîpati followed Brahmagupta here, completely disregarding the warning of Âryabhaṭa II, that "the mathematician who wishes to tell the area or the altitudes of a quadrilateral without knowing a diagonal is either a fool or a blunderer" (xv, 70).

Śrîpati's second rule for the area of a quadrilateral is remarkable for its contradiction of the corresponding rules of Śrîdhara and Mahâvîra; it does not appear in Brahmagupta's work. The rule states that "half the sum of the base and the face (of a quadrilateral) is multiplied by the perpendicular; the result is that which is [the measure] in the case of all [the plane figures] called quadrilateral" (xiii, 30). Earlier, Mahâvîra had given the same rule, noting that it does not hold in the case of a scalene quadrilateral (vii, 50). Śrîdhara, in his *Pâṭîgaṇita* (Rule 115), asserted that the rule is for finding the area of a quadrilateral with equal altitudes.

It is quite clear that Śrîpati's use of the term quadrilateral (caturbhuja) was not as consistent as that of Brahmagupta. Thus, his theorems on the quadrilaterals are applicable, respectively, to a quadrilateral in general (xiii, 26–27), a cyclic quadrilateral (xiii, 28), a quadrilateral with equal altitudes (xiii, 30), an isosceles trapezoid, and a square (xiii, 33). Elsewhere (xiii, 4), he used the term quadrilateral (caturbhuja) to mean a square (samacaturbhuja). Śrîpati simply dropped the appropriate prefixes. It may be noted that Âryabhaṭa I also did this by using the term caturbhuja in the sense of samacaturbhuja (ii, 5).

Śrîpati gave the following rule for finding the approximate square root of surd numbers:

> That, which results as the nearest square-root of the product of the numerator and denominator multiplied by ten thousand times any assumed square number, should be divided by the product of the root of the multiplier and the denominator; [the quotient is] the approximate [square] root. (xiii, 36)

This rule first appeared in the works of Śrîdhara (*Pâṭîgaṇita*, Rule 118) and was restated by Âryabhaṭa II (xv, 55) and later authors.

Śrîpati's rule for an arc and the corresponding sagitta is as follows:

> The [square] root extracted from six-times the square of the sagitta plus the square of the chord is here the arc. That which is the square-root [extracted] from the difference between the squares of the chord and the arc [which is] divided by six is the measure of the sagitta. (xiii, 39)

In modern mathematical notation, $s = \sqrt{c^2 + 6s^2}$ and $s = \sqrt{(a^2 - c^2)/6}$, where c is the chord, a the arc, and s the sagitta. It is easily seen that the second rule may be deduced from the first.

Though it is believed that these rules were known to the Jaina canonical writers, allegedly of the third century B.C. [Srinivasienger 1967, 22], Mahâvîra is the first Indian author to state them explicitly (vii, 73½). Besides giving these rules, Âryabhaṭa II gave an improved rule for the arc (xv, 98): a (neat) $= \sqrt{c^2 + (288/49)s^2}$, where the word "neat" seems to indicate the accurate value. Śrîpati apparently did not notice this rule.

Śrîpati gave a rule for the volume of the frustum of a rectangular pyramid turned upside down: "The aggregate of the distinct areas, resulting from the sum and from the plane areas, being divided by six and multiplied by the depth is the very accurate computation called volume" (xiii, 44).

The rule may be interpreted as follows (see *Lîlâvatî* [Apte 1937], vs. 217): Let a and b be the length and breadth of the face of the excavation, a^1 and b^1 are the length and breadth of its base, and h is the depth of the excavation. Then, the area resulting from the sum is $(a + a^1)(b + b^1)$, the area resulting from the plane areas is $ab + a^1b^1$, and the volume is $\{((a + a^1)(b + b^1) + (ab + a^1b^1))/6\} \cdot h$. Âryabhaṭa II gave the same rule (xv, 106), while Brahmagupta gave a slightly different one (xii, 45–46): approximate volume $= \{[(a + a^1)/2][(b + b^1)/2]\} \cdot h$; gross volume $= \{(ab + a^1b^1)/2\} \cdot h$; accurate volume = (gross volume − approximate volume)/3 + approximate volume. Brahmagupta thus arrived at the correct formula for the volume of the frustum of a rectangular pyramid by manipulating the approximate volume and the gross volume. Mahâvîra's procedure was essentially the same, although he used different nomenclature for the approximate volume and the gross volume (viii, 9–11½).

For the volume of a sphere Śrîpati's rule is: "The cube of the diameter of a stone sphere is divided by two; [the quotient], being increased by one-eighteenth of itself, is [its] volume; the stone measure [is] as before" (xiii, 46). If we take $\pi = 19/6$, this becomes the standard formula for the volume of a sphere. Śrîdhara gave the same rule in his Triśatî (Rule 56), but Âryabhaṭa I gave the inaccurate formula (ii, 7), (area of central circle)$^{3/2}$, probably suggested by the formula, (area of a face)$^{3/2}$, for the volume of a cube. Mahâvîra's accurate formula for the volume of a sphere is $(9/2) \cdot (9/10) \cdot (\text{radius}^3)$; (viii, $28\frac{1}{2}$). Brahmagupta did not give a formula for the volume of a sphere.

THE *AVYAKTAGAṆITÂDHYÂVA*

The algebraic chapter (Chap. xiv) of the *Siddhântaśekhara* consists of 37 verses, all embodying rules. Again, the manuscript evidence available does not permit the editor to establish the correct text for all the verses; thus five of these (4, 34–37) are too obscure to be interpreted and two (18, 31) are incomplete. The first verse (xiv, 1) is introductory and relates the divisions of algebra that are to be mastered for winning superiority among astrologers. The order and content of this chapter of the *Siddhântaśekhara* are almost the same as those of the Kuṭṭakâdhyâya of the *Brâhmasphuṭasiddhânta*. The avyaktagaṇitâdhyâya, however, contains some original sûtras concerning the symbols of algebra, signed numbers, operations on zero, surds, the solution of the simple linear equations, the solution of factums, the solution of the pulverizer and that of the square-nature, and the factorization of a given number. It does not contain any verses on the fundamental operations treated by Brahmagupta (xviii, 41–42).

Śrîpati's sûtra for the symbols is as follows:

> Colours such as "yâvattâvat" [so much as], black, blue, etc. should be assumed in the measure of the unknown; equally prominent with them are [the symbols in regard to] the growth of wealth etc. There should be a square [i.e., its symbol], a factum [i.e., its symbol], the measure of an arc [i.e., its symbol]. (xiv, 2)

Here, besides mentioning the colors as symbols for the unknowns, as did Brahmagupta (xviii, 2, 42, 51, etc.), Śrîpati hinted at the use of other symbols.

In addition to giving the rules stated by Brahmagupta (xviii, 30–35) for the laws of signs, Śrîpati gave the following rule in the fifth verse of the avyaktagaṇita:

> The two-squares, [that] of a negative and [that of] a positive (number), are positive; in their square-roots, they will become so. A negative [number] has no root because of its being non-square. In the same way, the operations on cubes should be performed.

Here, unlike Brahmagupta, Śrîpati gave rules for imaginary quantities and cubes of signed numbers. A similar reference to the square root of negative quantities was made by Mahâvîra earlier (i, 52). Thus Śrîpati and Mahâvîra were the earliest authors to recognize quite explicitly imaginary quantities. Śrîpati was the only Indian author to deal with the cubes of signed numbers. Though his rule is very brief, the indicated operations may be reconstructed very easily.

On operations with zero, the author gave the following rule:

> Positive and negative numbers do not undergo any change through the addition or subtraction of zero. Being subtracted from zero, positive [becomes] negative, negative [becomes] positive. On division [of positive and negative numbers] by zero, khahara [results]. (xiv, 6)

Śrîpati (like Brahmagupta) was not consistent regarding division of a nonzero number by zero. Here he put it as the indeterminate form "khahara," which is probably equivalent to the "taccheda" of Brahmagupta (xviii, 36); whereas in his arithmetic *Gaṇitatilaka* [Kapadia 1937, 29] he stated that this quotient is zero. Śrîpati did not explain what he meant by khahara, whereas Brahmagupta defined the teccheda (in modern notation) to be $(\pm a) \div 0 = \pm a/0$ or $0/\pm a$.

Śrîpati's treatment of surds begins with the following verse:

> The number whose square-root is not knowable is called a karaṇî. The divisor or multiplier of a karaṇî has been ascertained by scholars to be a square. (xiv, 7)

Here Śrîpati has given a confusing definition of a surd, according to which any number that does not yield a known root is a surd. If the definition were accepted, all numbers such as 2, 3, 5, 6, etc., would be surds. In fact Śrîpati has alluded here to the usual Indian concept [Colebrooke 1817, 145], that a surd is a perpetual square placed under the radical sign. The same concept is evident in his rule for the multiplier or divisor of a surd: Let $\sqrt{x}$ be a surd and let it be multiplied or divided by a rational number p. Then $\sqrt{x} \cdot p = \sqrt{xp^2}$ and $\sqrt{x}/p = \sqrt{x/p^2}$.

Śrîpati differed from Brahmagupta in only one rule for operating with surds, namely, in the addition and subtraction of surds. Śrîpati's rule is:

> In addition and subtraction you should multiply [each] karaṇî by an assumed quantity such that it becomes a square. You should divide the two squares of the sum and difference of the roots by the assumed multiplier. (xiv, 8)

Brahmagupta offered a slightly different rule:

> The surds being divided by an assumed quantity, and the square-roots of the quotients being extracted, the square of the sum of the roots, being multiplied by the assumed quantity, [is the sum], or the square of their difference [so multiplied, is the difference of the surds]. (xviii, 39)

In the vyaktagaṇita (xiii, 36), Śrîpati gave a rule equivalent to $\sqrt{a/b} = \sqrt{ab \cdot 10000x^2}/(b \cdot 100x)$, where x is an assumed multiplier for finding the approximate value of a surd. Brahmagupta did not give this rule.

Śrîpati's rule for the solution of the simple linear equation is:

> The difference of the absolute numbers being taken in the reverse order and divided by the difference [of the coefficients] of the unknown, [the two sides of the equality] should be the measures of the unknown. Alternately, if one wishes [one side] to be added with, decreased by, multiplied, or divided [by an appropriate number], so also is [it] to be performed on the other side. (xiv, 14)

The first part of the sûtra gives a rule known to Âryabhaṭa I (ii, 30) and explicitly stated by Brahmagupta (xviii, 44). But the second part gives a rule that embodies the use of the method of inversion to solve simple linear equations. Brahmagupta did not state this rule either.

Śrîpati's rules for the solution of the factum are:

> You should remove the factum from one side, the [simple] unknowns and the absolute numbers from the other. The product of the coefficients of the unknowns is added to the product of the absolute quantity and the coefficient of the factum. [The sum] being divided by an optional number, the quotient and the divisor are to be added arbitrarily to the greater or

smaller of the coefficients of the unknowns. [These two] divided by the coefficient of the factum will be the values of the unknowns in the reverse order. You should perform this operation [also] by the assumed values of the respective unknowns. (xiv, 20–21)

This is equivalent to the following procedure: (1) Let the equation be written as $axy = bx + cy + d$. Then the solutions of the equation will be given by $x = (m + c)/a$, $y = (ad + bc)/m + (b/a)$ or $x = (ad + bc)/m + (c/a)$, $y = (m + b)/a$, where m is arbitrary. (2) The second rule consists in the reduction of the given indeterminate equation to a simple one by assuming arbitrary values for all the unknowns except one.

Brahmagupta expressed the solution (xviii, 60) as

$$x = (m + c)/a, \quad y = (ad + bc)/m + (b/a),$$

where m is arbitrary and it is assumed that $b > c$ and $m > (ad + bc)/m$. If these conditions are reversed, then x and y will have their values interchanged. Brahmagupta also gave Śrîpati's second rule in a more explicit form, preferring it to the first rule (xviii, 62–63). Śrîpati's first rule is superior to that of Brahmagupta since it is free from the unnecessary restrictions imposed by the latter. But this rule cannot be attributed to him, as it had earlier been used by the author of the *Bakshâlî Manuscript* to solve the equation $xy = 3x + 4y \pm 1$ (folio 27, recto).

Śrîpati gave the following rule for the pulverizer:

Having divided the dividend, divisor and interpolator by their [greatest] common divisor, if any, divide the dividend and the divisor [abridged in that way] reciprocally until the residue is unity. Set down the quotients one below the other in succession; then underneath [the last] an optional number, and under it the corresponding quotient. [The optional number is determined as follows.] This [i.e., the last residue] is multiplied by some [chosen number] such that [the product], being increased or decreased [by the interpolator according as it is negative or positive], and [the sum or difference being] divided by the divisor [corresponding to the last residue] leaves no remainder. [That multiplier is the optional number.] [It is to be so] when the number of quotients is even. In the case of an odd number [of quotients], the interpolator, [if] negative, [must be first made] positive, and [if] positive, [must first be made] negative; the learned in this [branch of mathematics] say [so]. After determination of the optional number and multiplication of the number above it by it, you should add the quotient [corresponding to that optional number] there [i.e., to the product]. Proceeding upward, such an operation is to be performed again and again until two numbers are obtained. The first one being divided by the divisor, [the residue] should become [the least value of] the multiplier. Similarly, the second one, [being divided] by the dividend, [will give the least value of] the quotient [as the residue]. (xiv, 22–25) [1]

It should be noted that in obtaining the solution of the equation to be $ax = \pm c$, Śrîpati advised working out the mutual division to the end, i.e., until the last remainder is 1. Then, when the interpolator is positive, the chain is taken to be the sequence of quotients, with c and 0 attached at the end. When, however, the interpolator is negative, the chain is taken to be the sequence of quotients, with 0 and c divided by the last divisor attached at the end. By last divisor is meant the divisor corresponding to the last residue in the actual division of a and b. So, Śrîpati's rule is the same as that of Bhâskara II in his *Bîjagaṇita* [Dvivedi & Jha 1927, 25f.], and *Lîlâvatî* [Apte 1937, 252–254]. Śrîpati showed how to obtain the latter integral solutions in a way rather different from those of previous Indian authors.

In the following sûtra, Śrîpati showed how to determine whether a proposed problem in the pulverizer is solvable or not: "If the dividend and the divisor have a common divisor which is not a divisor of the interpolator, then it [i.e., the problem] is absurd" (xiv, 26). The rule is not found in any earlier Indian algebra, but is repeated by Bhâskara in his *Bîjagaṇita* [Dvivedi & Jha 1927, 24f.] and *Lîlâvatî* [Abte 1937] (vv. 242–243).

Śrîpati also showed how to find all positive integral solutions of a Kuṭṭaka problem with the help of its least integral solutions.

> The two [i.e., the dividend and the divisor] being divided reciprocally, the last residue will be their [greatest] common divisor. The divisor and the dividend are each divided by it and [the quotients are each] multiplied by an optional number. Add [the products] respectively to the multiplier and the quotient [in order to obtain multiple sets of values of these two]. (xiv, 27)

This rule may be interpreted as follows: Adding iab, where i is an arbitrary number, to both sides of the equation $by = ax \pm c$, we get $b(y + ia) = a(x + ib) \pm c$. Hence the multiplier $= x + ib$ and quotient $= y + ia$. Here we substitute $i = k/g$, where g is the greatest common divisor of a and b, thus obtaining multiplier $= x + kb'$ and quotient $= y + ka'$, where $a' = a/g$, $b' = b/g$. Hence the rule.

Śrîpati's treatment of the vargaprakṛti begins with the following condition for integral roots of the auxiliary equation:

> If 2, 1, or 4 be the additive or subtractive [of the auxiliary equation], the lesser and greater roots should be integral. (xiv, 32)

According to this rule, if ± 1, ± 2, or ± 4 be the interpolator of the auxiliary equation $Na^2 + k = b^2$, then the lesser and greater roots of the equation $Nx^2 + 1 = y^2$ will be integral. This lemma had been earlier stated by Brahmagupta for the value ± 4 of the interpolator (xviii, 67–68).

Śrîpati then described a method to obtain rational solutions of the vargaprakṛti:

> Unity is the lesser root. Its two squares [set at two places] are [each] multiplied by the prakṛti [and the product is] decreased [by the prakṛti and] increased by a [suitable] interpolator. That which is the [square] root [of the interpolator] is the greater root. From these two, two roots are obtained by bhâvanâ. [There will] also [be] an infinite set of two [roots]. (xiv, 33) [2]

Equivalently, let 1 be assumed as the lesser root. Then we get the identities $N \cdot 1^2 + (m^2 - N) = m^2$ and $N \cdot 1^2 + (m^2 - N) = m^2$. Applying Brahmagupta's bhâvanâ to these identities, we easily obtain

$$x = 2m/(m^2 - N), \qquad y = (m^2 + N)/(m^2 - N),$$

where m is any rational number, as a rational solution of the equation $Nx^2 + 1 = y^2$. Also, an infinite set of such solutions is obtained by manipulating the value of m^2.

Śrîpati then gave the following rules for the integral solutions of the vargaprakṛti:

> For finding integral roots, divide or multiply [the prakṛti] by the square of an optional number. The prakṛti [so transformed] here is then increased or decreased by an interpolator so that it

becomes a square. Then [extract] the square root. [The prakṛti so transformed] added to the square of the greater root [will give a fresh greater root]. The [obtained] interpolator is square [of its previous value]. The greater and lesser roots which are smaller [in the equation] with additive unity will be obtained on division of the greater and the lesser root by the square-root [of the interpolator]. Again, the additive or subtractive being equal to unity, the square of the lesser root is multiplied by the prakṛti [in order to obtain the greater root]. (xiv, 34–35) [3]

These two verses mean: Let a be an arbitrary integral number such that $Na^2 + k = b^2$, where b is another integral number. Then by the principle of bhâvanâ, $N(2ab)^2 + k^2 = (Na^2 + b^2)^2$. Hence $2ab/k$ and $(Na^2 + b^2)/k$ will be a solution of the equation $Nx^2 + 1 = y^2$. Under the same rule, the author has instructed that the same operations may be performed with the help of a divisor instead of a multiplier, so that the process may be executed with the help of the auxiliary equation $N/a^2 + k = b^2$. Under the same verses, the author has also suggested that if a be an assumed integer such that $Na^2 \pm 1$ is a perfect square, say b^2, where b is integral, then (a, b) is a solution of $Nx^2 \pm 1 = y^2$. The first two rules are simple consequences of Brahmagupta's bhâvanâ (xviii, 64–65).

Śrîpati, like other Indian authors, said nothing about the existence of a solution of the vargaprakṛti and the completeness of those solutions exhibited in the case of integral roots. Further, instead of using Brahmagupta's terms (xviii, 64, 66f.) "first root" for the value of x and "last root" for the value of y, which are free from ambiguity, Śrîpati used the commentator's terms, "lesser root" and "greater root," respectively, though they do not appear to be always accurate and fortunate.

For the factorization of a given number, Śrîpati gave the following rules:

If the dividend is even, it is to be divided by 2; if 5 occurs in its first place, [it is to be divided] by 5. This should be done repeatedly until the dividend becomes odd [with units digit different from 5]. Also, divide the dividend [which is not even, nor is there 5 in its units place] by the divisors 3, etc. [i.e., the prime numbers 3, 7, 11 etc.]. If the dividend is a perfect square, its square-root itself is its divisor; if not, its nearest square-root should be multiplied by 2, then increased by 1, and then diminished by the residue of the square-root; in case it [i.e., the resulting number] be a perfect square, [find] its square-root and also the square-root [of the dividend] as increased by this [perfect square]; [and take their] sum and difference [which will be the two factors of the dividend]. (xiv, 36–37) [4]

Rule 1 (contained in the first verse) gives the ordinary method. In mathematical terminology, rule 2 is as follows: Let N be a square equal, say, to a^2; then its square root, a, will be a divisor or factor. Let N^1 be a non-square number, say, $N^1 = a^2 + r$; then if $2a + 1 - r = b^2$, we have $N^1 + b^2 = a^2 + r + (2a + 1 - r) = (a + 1)^2$; so, $N^1 = (a + 1)^2 - b^2 = (a + 1 + b)(a + 1 - b)$. It is to be noted that this rule fails in the case where $2a + 1 - r$ is not a perfect square.

All currently available evidence suggests that Śrîpati was the earliest Indian author to give such rules. Later Nârâyaṇa dealt with this topic more exhaustively in his *Gaṇitakaumudî* (II, Chap. XI, Rules 2–9).

ŚRÎPATI'S SOURCES

As pointed out earlier, Śrîpati was certainly familiar with the works of Śrîdhara. Moreover, he was undoubtedly familiar with the mathematical works of Brahmagupta, for he repeatedly mentioned Brahmagupta's name in the *Siddhântaśekhara* (e.g., ii, 58; and xviii, 18). Śrîpati also reproduced one arithmetical rule of the *Brâhmasphuṭasiddhânta* (xii, 14, first hemistich) word for word in his *Gaṇitatilaka* [Kapadia 1937, 85].

It is not clear whether the name Âryabhaṭa mentioned by Śrîpati refers to Âryabhaṭa I or Âryabhaṭa II. It may be pointed out that the rule for squaring and cubing in the *Siddhântaśekhara* (xiii, 4) resembles the corresponding rule in the *Âryabhaṭîya* (ii, 3); moreover, in the *Siddhântaśekhara* (xv, 15–17), Śrîpati contradicted Âryabhaṭa I's doctrine of the movement of the earth. But he could have learned of Âryabhaṭa's theory of the earth's rotation from the *Brâhmasphuṭasiddhânta* with Pṛthudaka's vivaraṇa. It is to be noted that the rules for division (xiii, 3) and for the determination of the volume of an excavation (xiii 44), found in the *Siddhântaśekhara* resemble the corresponding rules in the *Mahâsiddhânta*. On the other hand, Śrîpati ignored the following rules of Âryabhaṭa II: the rule for the surface of a sphere (xv, 93–94); an improved rule for the measure of an arc of a circle (described above); and an improved rule for ekapatrîkaraṇa (mentioned before). Śrîpati also ignored Âryabhaṭa II's bitter criticism of the formula $\sqrt{(s-a)(s-b)(s-c)(s-d)}$ for the area of a quadrilateral (cited above). But he could well have ignored these in following the tradition of Brahmagupta and Śrîdhara; there is insufficient evidence to conclude that Śrîpati was ignorant of the *Mahâsiddhânta*.

Śrîpati's inclusion of the garland problem in the *Gaṇitatilaka* [Kapadia 1937, 6] and his treatment of the linear, quadratic, and radical equations there [Kapadia 1937, 41–64] remind us of the *Gaṇitasârasaṁgraha*. Śrîpati's rule for the sum of a series in geometric progression (*Siddhântaśekhara*, xiii, 25) and his nomenclature for the vargaprakṛti (*Siddhântaśekhara*, xiv, 32–35) resemble those given by Pṛthudakaswâmî. So, in general, it may be said that Śrîpati seems to have known the works of Mahâvîra and Pṛthudakaswâmî as well.

Śrîpati's methods of addition and subtraction of whole numbers [Kapadia 1937, 3–4], as explained by his commentator, differ from the Indian methods reported by B. Datta and A. N. Singh [Datta & Singh 1962, 130–133]. Instead, Śrîpati's method of addition resembles the angular addition of the present day. As these rules do not appear in any earlier Indian work, their sources cannot be identified.

GLOSSARY

Dhîkoṭidakaraṇa	An astronomical work by Śrîpati that deals with solar and lunar eclipses.
Jyautiṣaratnamâlâ Dhruvamânasa Jâtakapaddhati	Astrological works by Śrîpati.
Brahmin	A caste of Indian Hindus.
Kâśyapa	Descendants of the sage kaśyapa.
bhâga	Combination of fractions of the type $(a/b) \pm (c/d) \pm (e/f) \pm \ldots$, where $a, b, c, \ldots$ are integers.
prabhâga	Combination of fractions of the type (a/b) of (c/d) of $(e/f) \ldots$.
bhâgânubandha	Combination of fractions of the type $a + (b/c)$ or $(p/q) + (r/s)$ of $(p/q) + \ldots$.

bhâgâpavâha	Combination of fractions of the type $a - (b/c)$ or $(p/q) - (r/s)$ of $(p/q) + \ldots$.
vallîsavarṇana	Reduction of a combination of fractions in continued proportion into a proper fraction.
bhâga-bhâga	Combination of fractions of the type $a \div (b/c)$ or $(a/b) \div (c/d)$.
bhâgamâtṛ	Combination of two or more of the types bhâga, prabhâga, bhâgânubandha, bhâgapavâha and bhâgabhâga.
dṛśyajâti	Problems wherein is given the numerical value of the portion remaining after removing certain specified fractional parts of the total quantity to be found.
śeṣajâti	Problems wherein the numerical value is given of the portion remaining after removing a known fractional part of the total quantity to be found, as also after removing certain known fractional parts of the successive remainders.
viśleṣajâti	Problems wherein the numerical value is given of the portion remaining after removing a known fractional part of the total quantity, then another fractional part of the total quantity, then a multiple or submultiple of the difference between these two fractional parts, and so on.
śeṣamûlajâti	Problems wherein the numerical value is given of the portion remaining after subtracting from the total quantity certain fractional parts thereof, as also a multiple of the square root of the remainder.
mûlâgrabhâgajâti	Problems wherein the numerical value is given of the portion remaining after subtracting from the total quantity certain fractional parts thereof, as also a multiple of the square root of the total quantity.
ubhayâgradṛśyajâti	Problems wherein a known number of things is first removed, then some fractional parts of the successive remainders, and then some multiple of the square root of the total quantity are removed; and last, the numerical value of the remaining portion is given.
bhinnabhâgadṛśyajâti	Problems wherein a fractional part of the whole multiplied by another fractional part thereof is removed from it, and the remaining portion is expressed as a fraction of the whole.
bhâgamûlajâti	Problems wherein a multiple of the square root of a fractional part of the total number is first removed, and then the numerical value of the remaining portion is given.
hîṇavargajâti	Problems wherein the numerical value is given of the remainder after removing from the whole the square of a fractional part thereof, the fractional part being at the same time decreased by a given number.
Hemistich	Half-verse.
Factum	Quadratic indeterminate equations of the type $axy = bx + cy + d$, where x and y are unknowns.
Pulverizer	A branch of algebra dealing with indeterminate equations of the first degree.
Kuṭṭaka	Pulverizer.

NOTES

1. B. Datta and A. N. Singh have given a defective translation of Śrîpati's rule on the pulverizer and have thus equated it to that of Bhâskara I [Datta & Singh 1962, 110].

2. Babuaji Misra has given a different interpretation of Śrîpati's rule (xiv, 33) in his commentary:

> Unity or any integral number is assumed as the lesser root. It is multiplied by the prakṛti and the product is decreased or increased by a kṣipti. That which is the square-root of the resulting number will be the greater root. From these lesser and greater roots, an infinite set of lesser and greater roots will be obtained by bhâvanâ.

Here Babuaji has interpreted rûpam as "any integer." But rûpam means unity in mathematics. Also the rule meant by Babuaji was mentioned by Śrîpati in the two verses that follow this verse (xiv, 34–35). There is no justification for merely duplicating the same rule.

3. Read chidyât for chindyât in the first hemistich, jyeṣṭhavargena for jyeṣṭhavadhena in second hemistich of the first verse, and tvalpavargaṁ for tvalpapadaṁ in the second hemistich of the second verse.

4. Read dvâbhâyam for tâbhyâm in the first hemistich of the first verse, and vargaścet for vargaiścet in the first hemistich of the second verse.

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